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On the Legendre transformation for singular Lagrangians and related topics

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Abstract. We study the concept of projectability by the Legendre transformation $FL: TQ \rightarrow T^*Q$ induced by a singular Lagrangian, and the restrictions on this concept when submanifolds of TQ are considered. The FL projectability of the characteristic vector fields and constraint functions relative to such a submanifold is analysed, thus giving a way of characterising the submanifolds of TQ according to their behaviour under FL, by means of the study of the tangency of the fields belonging to ker FL_*. The application of some of the results to the theory of constrained systems is discussed.

1. Introduction

The study of constrained dynamical systems has been a matter of increasing interest in recent years. Many authors have contributed to the development of the Hamiltonian formalism for such systems [1-3]. Nevertheless, some of the most significant advances have been made recently in the analysis of the Lagrangian formalism, as well as in the equivalence between both formulations [4-7]. The essential feature to be pointed out in this formalism is the singular character of the Lagrangian function.

One of the main consequences of this property of the Lagrangian is that the Legendre transformation (fibre derivative of the Lagrangian, FL, in geometrical language [8]) is not a diffeomorphism. In these cases, to prove the equivalence with the corresponding Hamiltonian formulation is not an easy task, but this problem has already been solved [4, 7]. Another question is that some Lagrangian p-forms or vector fields have no Hamiltonian counterpart. This 'FL projectability' question arises, for instance, when constraint algorithms for the Lagrangian equations of motion are studied. In these cases, some constraints having no Hamiltonian counterpart appear [7]. Although the mathematical origin of this kind of constraint function is known [9], the role they play in the geometrical structure of the Lagrangian systems is still under investigation [10].

The aim of this paper is to contribute to this investigation, by studying the projectability of tensorial objects by FL. In order to apply it to the constraint algorithms, we pay special attention to investigate it in relation to submanifolds of the tangent bundle TQ, where the Lagrangian formalism is developed (§ 2). We also study the behaviour of typical vector fields in these submanifolds under the action of FL_* (§ 3), as well as the geometrical meaning of the presence of non-FL-projectable constraints (§ 4). Finally, the constraint algorithms are studied in this framework (§ 5).

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2. The concept of projectability by the Legendre transformation

The Lagrangian formulation of mechanics is performed geometrically in the tangent bundle TQ of a differential manifold Q (configuration space). Given a function $L \in \Lambda^0(TQ)$ we construct the Lagrange 2-form $\omega_L \in \Lambda^2(TQ)$ and the energy function $E_L \in \Lambda^0(TQ)$, which allows us to write the Lagrangian equations of motion in the form

$$i(X_L)\omega_L - \mathrm{d}E_L = 0 \tag{1}$$

($i(X_L)\omega_L$ denotes the contraction of the vector field X_L with the form ω_L). The function L is called the Lagrangian function and (TQ, ω_L, E_L) is its associated Lagrangian system [11, 12] (the only condition for L to be a good function to describe the dynamics is that rank ω_L be constant on TQ, locally at least, and we assume that this is so).

It is well known that, given a Lagrangian function L, the connection between the Lagrangian and Hamiltonian formalisms is performed by the fibre derivative of L, $FL: TQ \to T^*Q$, which is usually called the Legendre transformation. We can also note that the only case in which FL is a local or global diffeomorphism corresponds to the so-called regular or hyperregular Lagrangians (ω_L is a symplectic form), and the fact that FL is not a diffeomorphism is equivalent to L being a degenerate or singular Lagrangian function and ω_L being presymplectic. In this last case we will assume FL to be a submersion from TQ onto its image and the fibres $FL^{-1}(FL(x))$, $\forall x \in TQ$, to be connected submanifolds of TQ. Systems verifying these conditions are called almost Lagrangian systems [4], and it also can be proved that these assumptions are sufficient conditions for a degenerate Lagrangian system to have an equivalent Hamiltonian formulation [4].

When we deal with regular Lagrangian systems, FL is a diffeomorphism and so are the induced maps FL_* and FL^*; therefore every vector field or p-form in TQ has a similar counterpart in T^*Q. If the Lagrangian system is singular this is not always true because FL is not a diffeomorphism. Locally, this means that it is not possible to isolate all the coordinates (v^i) of a local chart in TQ as functions of the natural symplectic coordinates (q^i, p_i) of T^*Q. Therefore, any magnitude in TQ having explicit dependence on these coordinates does not have a canonical expression in T^*Q. Due to this, we make the following definition.

Definition 2.1. (*a*) A function $f \in \Lambda^0(TQ)$ (resp a differential p-form $\alpha \in \Lambda^p(TQ)$) is said to be FL projectable iff $\exists f' \in \Lambda^0(T^*Q)$ such that $FL^*f' = f$ (resp $\exists \alpha' \in \Lambda^p(T^*Q)$ such that $FL^*\alpha' = \alpha$).

(*b*) A vector field $X \in \mathscr{X}(TQ)$ is said to be FL projectable iff $\exists X' \in \mathscr{X}(T^*Q)$ such that $FL_*X = X'$. This is equivalent to the following statement: $\exists X' \in \mathscr{X}(T^*Q)$ such that $X(FL^*f') = FL^*(X'(f'))$, $\forall f' \in \Lambda^0(T^*Q)$; and then $X' = FL_*X$.

Notice that definition (*b*) is also equivalent to demanding that for every FL-projectable function f or p-form α, $X(f)$ and $\pounds(X)\alpha$ (Lie derivative) are a FL-projectable function or p-form, respectively. We introduce the notation $\Lambda^0(TQ)_{FL}$, $\Lambda^p(TQ)_{FL}$ and $\mathscr{X}(TQ)_{FL}$ to denote the sets of FL-projectable functions, p-forms and vector fields, respectively.

It is also easy to prove that if X_1, $X_2 \in \mathscr{X}(TQ)_{FL}$, then $[X_1, X_2] \in \mathscr{X}(TQ)_{FL}$ and $FL_*[X_1, X_2] = [FL_*X_1, FL_*X_2]$.

Consider now the submanifold $M_0 \equiv FL(TQ)$ (which in the Hamiltonian formalism is called the primary constraint submanifold) and its embedding $j_0': M_0 \hookrightarrow T^*Q$. Let $FL_0: TQ \to M_0$ be the submersion implicitly defined by $FL = j_0' \circ FL_0$. Then, if

$C^0(T^*Q, M_0)$ denotes the ideal of functions in T^*Q vanishing on M_0, since $FL^*f' = FL_0^* j_0'^* f'$, $\forall f' \in \Lambda^0(T^*Q)$, it is evident that $FL^*f' = 0$ if and only if $f' \in C^0(T^*Q, M_0)$. Hence, $\forall X \in \mathscr{X}(TQ)_{FL}$ and $\forall \zeta' \in C^0(T^*Q, M_0)$, we have

$$0 = X(FL^*\zeta') = FL^*(FL_*X(\zeta')) = FL_0^* j_0'^*(X'(\zeta')) \Leftrightarrow j_0'^*(X'(\zeta')) = 0$$

and so $X' \in \boldsymbol{\mathscr{X}}(M_0) \equiv \{Y' \in \mathscr{X}(T^*Q) \text{ tangent to } M_0\}$. Thus it is easy to prove the following.

Proposition 2.2. (i) $f \in \Lambda^0(TQ)_{FL}$ if and only if $\exists f_0' \in \Lambda^0(M_0)$ such that $f = FL_0^* f_0'$. Then any $f' = f_0' + \zeta' \in \Lambda^0(T^*Q)$ (where f_0' is an extension of f_0 to T^*Q and $\zeta' \in C^0(T^*Q, M_0)$) verifies $FL^*f' = f$.

(ii) $X \in \mathscr{X}(TQ)_{FL}$ if and only if $\exists X_0' \in \mathscr{X}(M_0)$ such that $FL_{0*}X = X_0'$ or, equivalently, $\exists X_0' \in \mathscr{X}(M_0)$ such that $X(FL_0^* f_0') = FL_0^*\ (X_0'(f_0'))$, $\forall f_0' \in \Lambda^0(M_0)$ (and then $X_0' = FL_{0*}X$).

A characterisation of the vector fields belonging to ker FL_* is performed by means of the FL-projectable functions, by saying that $\Gamma \in \ker FL_*$ if and only if $\Gamma(f) = 0$, $\forall f \in \Lambda^0(TQ)_{FL}$. Nevertheless, it is more interesting in the converse sense, i.e. to check the character in relation to the FL projectability of functions, p-forms and vector fields. Thus, taking into account the comments in the appendix and known results from the theory of reduction [8, 13] one can prove the following.

Theorem 2.3. (i) $f \in \Lambda^0(TQ)_{FL}$ (resp $\alpha \in \Lambda^p(TQ)_{FL}$) if and only if f (resp α) is τ_0 projectable or, equivalently, $\Gamma(f) = 0$ (resp $\pounds(\Gamma)\alpha = 0$), $\forall \Gamma \in \ker FL_*$.

(ii) $X \in \mathscr{X}(TQ)_{FL}$ if and only if X is τ_0 projectable or, equivalently, $[\Gamma, X] \in \ker FL_*$, $\forall \Gamma \in \ker FL_*$.

An interesting property arising from the last result is that one can also find a local basis of $\mathscr{X}(TQ)$ made up of vector fields (Γ_μ, Y_α), where (Γ_μ) is a local basis of ker FL_* and $(Y_\alpha) \in \mathscr{X}(TQ)_{FL}$ (and hence $(\tau_{0*}Y_\alpha)$ is a local basis of $\mathscr{X}(\mathscr{S}_0)$ on its turn). Therefore, any vector field $X \in \mathscr{X}(TQ)$ may be written as $X = f^\mu \Gamma_\mu + f^\alpha Y_\alpha$ with f^μ, $f^\alpha \in \Lambda^0(TQ)_{FL}$ and the necessary and sufficient condition for $X \in \mathscr{X}(TQ)_{FL}$ is $f^\alpha \in \Lambda^0(TQ)_{FL}$ since

$$[\Gamma, X] = \Gamma(f^\mu)\Gamma_\mu + f^\mu[\Gamma, \Gamma_\mu] + \Gamma(f^\alpha)Y_\alpha + f^\alpha[\Gamma, Y_\alpha] \in \ker FL_* \Leftrightarrow \Gamma(f^\alpha) = 0.$$

Since we have in mind applications to the constraint theory, we consider now a submanifold $j_S : S \hookrightarrow TQ$ and $M_S \equiv FL(S)$. An obvious extension of the concept of FL projectability may be given as follows (see the appendix for notation).

Definition 2.4. (*a*) A function $f_S \in \Lambda^0(S)$ (resp a p-form $\alpha_S \in \Lambda^p(S)$) is said to be FL_S projectable iff $\exists f_S' \in \Lambda^0(M_S)$ such that $FL_S^* f_S' = f_S$ (resp $\exists \alpha_S' \in \Lambda^p(M_S)$ such that $FL_S^* \alpha_S' = \alpha_S$). A function $f \in \Lambda^0(TQ)$ (resp a p-form $\alpha \in \Lambda^p(TQ)$) is said to be weakly FL projectable relative to S iff its specialisation in S is FL_S projectable, i.e. $\exists f' \in \Lambda^0(T^*Q)$ such that $FL_S^* j_S'^* f' = j_S^* f$ (resp $\exists \alpha' \in \Lambda^p(T^*Q)$ such that $FL_S^* j_S'^* \alpha', = j_S^* \alpha$).

(*b*) A vector field $X_S \in \mathscr{X}(S)$ is said to be FL_S projectable iff $\exists X_S' \in \mathscr{X}(M_S)$ such that $FL_{S*}X_S = X_S'$ or, equivalently, iff $\exists X_S' \in \mathscr{X}(M_S)$ such that $X(FL_S^* f_S') = FL_S^*(X_S'(f_S'))$, $\forall f_S' \in \Lambda^0(M_S)$, and then $X_S' = FL_{S*}X$. A vector field $X \in \mathscr{X}(TQ)$ is said to be weakly FL projectable relative to S iff

(i) $X \in \boldsymbol{\mathscr{X}}(S)$ ($\boldsymbol{\mathscr{X}}(S)$ denotes the set of vector fields of $\mathscr{X}(TQ)$ which are tangent to S, satisfying $J_{S*}\mathscr{X}(S) = \boldsymbol{\mathscr{X}}(S)|_S$), and

(ii) the vector field $X_S \in \mathscr{X}(S)$ such that $j_{S*}X_S = X|_S$, is FL_S projectable or, equivalently, iff $\exists X' \in \mathscr{X}(M_S)$ such that

$$j_S^*(X(FL^*f')) = j_S^* FL^*(X'(f')) = FL_S^* j_S'^*(X'(f')) \qquad \forall f' \in \Lambda^0(T^*Q).$$

We will denote by $\Lambda^0(S)_{FL}$, $\Lambda^p(S)_{FL}$ and $\mathscr{X}(S)_{FL}$ the sets of FL_S-projectable functions, p-forms and vector fields, respectively, and $\Lambda^0(TQ, S)_{FL}$, $\Lambda^p(TQ, S)_{FL}$ and $\mathscr{X}(TQ, S)_{FL}$ the sets of those that are weakly FL projectable relative to S.

Some aspects to be pointed out are the following. First, the concept of FL projectability is recovered as a particular case of the preceding one by making $S = TQ$. Second, it is evident that $\mathscr{X}(TQ, S)_{FL} \supset \mathscr{X}(TQ)_{FL} \cap \mathscr{X}(S)$, whereas every FL-projectable function or p-form is also weakly FL projectable relative to any $S \hookrightarrow TQ$. Note that the converse is not true because if $f_S \in \Lambda^0(S)_{FL}$ and $f_S \in \Lambda^0(TQ, S)_{FL}$ is an extension of f_S in TQ, it suffices to take $f = f_S + \lambda^\mu \zeta_\mu$, with $\zeta_\mu \in C^0(TQ, S)$ and $\lambda^\mu \notin \Lambda^0(TQ)_{FL}$ arbitrary, to obtain a weakly FL-projectable but not FL_S-projectable function. Finally, notice that $\mathscr{X}(S)_{FL}$ is closed under the Lie bracket and

$$FL_{S*}[X_{S1}, X_{S2}] = [FL_{S*}X_{S1}, FL_{S*}X_{S2}] \qquad \forall X_{S1}, X_{S2} \in \mathscr{X}(S)_{FL}.$$

Vector fields in TQ and T^*Q can be classified as follows:

$\mathscr{X}(TQ)$	$\mathscr{X}(T^*Q)$
$X \notin \mathscr{X}(TQ)_{FL}$	—
$X \in \mathscr{X}(TQ)_{FL}$ but $X \notin (TQ, S)_{FL}$	$FL_*X \in \mathscr{X}(M_0)$ but $FL_*X \notin \mathscr{X}(M_S)$
$X \in \mathscr{X}(TQ)_{FL}$ and $X \in \mathscr{X}(TQ, S)_{FL}$	$FL_*X \in \mathscr{X}(M_0)$ and $FL_*X \in \mathscr{X}M_S)$
—	$X' \notin \mathscr{X}(M_0)$

In order to check the character of the tensorial objects in relation to their FL_S projectability and weakly FL projectability, we generalise theorem 2.3. Thus, taking into account the comments in paragraphs (a) and (b) of the appendix, we claim the following.

Theorem 2.5. Let S be a submanifold of TQ such that $\ker FL_* \cap \mathscr{X}(S) \neq \{0\}$ (and $\dim(\ker FL_* \cap \mathscr{X}(S))_x = \text{constant}$, $\forall x \in TQ$). Then

(i) the necessary and sufficient condition for $f_S \in \Lambda^0(S)_{FL}$ is that f_S be τ_S projectable, i.e. $\Gamma_S(f_S) = 0$ (resp $\alpha_S \in \Lambda^p(S)_{FL} \Leftrightarrow \pounds(\Gamma_S)\alpha_S = 0$), $\forall \Gamma_S \in \ker FL_{S*}$. As a consequence, $f \in \Lambda^0(TQ, S)_{FL}$ if and only if $j_S^*(\Gamma(f)) = 0$ (resp $\alpha \in \Lambda^p(TQ, S)_{FL} \Leftrightarrow j_S^* \mathscr{L}(\Gamma)\alpha = 0$), $\forall \Gamma \in \mathbf{ker}\, \boldsymbol{FL_*} \equiv \ker FL_* \cap \mathscr{X}(S))$, and

(ii) the necessary and sufficient condition for $X_S \in \mathscr{X}(S)_{FL}$ is that X_S be τ_S projectable, i.e. $[\Gamma_S, X_S] \in \ker FL_{S*}$. As a consequence, if $X \in \mathscr{X}(TQ)$, $X \in \mathscr{X}(TQ, S)_{FL}$ if and only if $X \in \mathscr{X}(S)$ and $[\Gamma, X] \in \mathbf{ker}\, \boldsymbol{FL_*}$, $\forall \Gamma \in \mathbf{ker}\, \boldsymbol{FL_*}$. (Notice the particular case in which $\ker FL_* \subset \mathscr{X}(S)$.)

Observe that, if $\ker FL_* \cap \mathscr{X}(S) = 0$, then the projection τ_S is the identity and every object is FL_S projectable (see the appendix).

Some important examples of FL-projectable objects are the Lagrange form ω_L and the energy function E_L, because $\omega_L = FL^*\Omega$ and $E_L = FL^*h$, where $\Omega \in \Lambda^2(T^*Q)$ is the natural symplectic form in T^*Q and $h \in \Lambda^0(T^*Q)$ is the Hamiltonian function (which always exists for almost regular Lagrangian systems [4]).

3. On the *FL* projectability of distinguished vector fields in a submanifold $S \hookrightarrow TQ$

If S is a submanifold of TQ, it is obvious that $M_S \equiv FL(S)$ is submanifold of $M_0 \hookrightarrow T_*Q$. Then $C^0(T^*Q, M_0) \subset C^0(T^*Q, M_S)$. We call the elements of $C^0(T^*Q, M_0)$ primary constraints and the rest of the elements of $C^0(T^*Q, M_S)$ secondary constraints.

Now we shall introduce some notation based on [4, 8, 14]. Let (P, ω) be a symplectic or presymplectic manifold and $j: S \hookrightarrow P$ a submanifold. $\mathscr{X}(S)^\perp$ denotes the set of vector fields in P whose restrictions on S take values in the orthogonal complement $TS^\perp$ of $Tj(TS)$ in $T(P)$, i.e.

$$\mathscr{X}(S)^\perp \equiv \{Z \in \mathscr{X}(P) / j^* i(Z)\omega = 0\}.$$

If $\omega_S = j_S^*\omega$, $\mathbf{ker}\ \boldsymbol{\omega}_S \in \mathscr{X}(P)$ is made up of the extensions in P of the vector fields belonging to $\ker \omega_S \subset \mathscr{X}(S)$, and we have $\mathbf{ker}\ \boldsymbol{\omega}_S = \mathscr{X}(S)^\perp \cap \boldsymbol{\mathscr{X}}(S)$. Putting these concepts into the present context and taking into account that $\omega_L = FL^*\Omega$, a simple computation allows us to prove the following results.

Proposition 3.1. Let (TQ, ω_L, E_L) be an almost regular system and $j_S: S \hookrightarrow TQ$ a submanifold such that $\ker FL_* \cap \boldsymbol{\mathscr{X}}(S) \neq \{0\}$. Then

(i) $\ker FL_* \subset \mathscr{X}(S)^\perp$, in particular, $\ker FL_* \subset \ker \omega_L \equiv \mathscr{X}(TQ)^\perp$,

(ii) $\ker FL_* \cap \boldsymbol{\mathscr{X}}(S) = \mathbf{ker}\ \boldsymbol{\omega}_{LS}$ (with $\omega_{LS} = j_S^*\omega_L$), and

(iii) $[\mathbf{ker}\ \boldsymbol{FL}_{S*}, \mathscr{X}(S)^\perp] \subset \mathscr{X}(S)^\perp$.

Theorem 3.2. With the hypothesis of proposition 3.1 we have the following.

(i) There exists a local basis of $\mathscr{X}(S)^\perp$ which only contains FL-projectable vector fields. Denoting this base by $\beta(\mathscr{X}(S)^\perp \cap \mathscr{X}(TQ)_{FL})$, we then have

$$FL_*\beta(\mathscr{X}(S)^\perp \cap \mathscr{X}(TQ)_{FL}) = \beta(\mathscr{X}(M_S)^\perp \cap \boldsymbol{\mathscr{X}}(M_0)).$$

(ii) There exists a local basis of $\mathbf{ker}\ \boldsymbol{\omega}_{LS}$ which only contains weakly FL-projectable vector fields. Then we have

$$FL_*\beta(\mathbf{ker}\ \boldsymbol{\omega}_{LS} \cap \mathscr{X}(TQ, S)_{FL}) = \beta(\mathbf{ker}\ \boldsymbol{\omega}'_S) \qquad (\boldsymbol{\omega}'_S = j'^*_S\Omega).$$

Proof. (i) Consider the quotient manifold $\mathscr{S}_0 \equiv TQ/\mathscr{F}_L^0$ and its submanifold $\mathscr{S}_S \equiv S/\mathscr{F}_L^S$ (see figure 1). Since ω_L is FL projectable, a presymplectic form $\tilde{\omega}_L \in \Lambda^2(\mathscr{S}_0)$ exists such that $\tau_0^*\tilde{\omega}_L = \omega_L$. On the other hand, $\forall \tilde{X} \in \mathscr{X}(\mathscr{S}_0)$, $\exists X \in \mathscr{X}(TQ)/\tau_{0*}X = \tilde{X}$. First, we shall prove that $\forall \tilde{X} \in \mathscr{X}(\mathscr{S}_S)^\perp$, the corresponding $X \in \mathscr{X}(TQ)$ belongs to $\mathscr{X}(S)^\perp$. In fact, since τ_S is a submersion we have

$$\begin{aligned}\tilde{X} \in \mathscr{X}(\mathscr{S}_S)^\perp \Leftrightarrow \tilde{j}_S^* i(\tilde{X})\tilde{\omega}_L = 0 \Rightarrow 0 = \tau_S^*\tilde{j}_S^* i(\tilde{X})\tilde{\omega}_L = j_S^*\tau_0^* i(\tilde{X})\tilde{\omega}_L \\ = j_S^* i(X)\omega_L \Leftrightarrow X \in \mathscr{X}(S)^\perp.\end{aligned}$$

Second, if we denote by $T\mathscr{F}_L^0$ the bundle spanned by the distribution $\ker FL_*$, taking into account proposition 3.1(i) and remembering that the vector fields of $\mathscr{X}(S)^\perp$ and $\mathscr{X}(\mathscr{S}_S)^\perp$ take values on the bundles $TS^\perp$ and $T\mathscr{S}_S^\perp$, respectively, one can prove that $TS^\perp/T\mathscr{F}_L^0 = T\mathscr{S}_S^\perp$. Therefore, $\forall x \in S$ and $\forall \mathscr{L}_{(x)} \in \mathscr{S}_S$, we have

$$\dim T_xS = \dim T_{\mathscr{L}(x)}\mathscr{S}_S + \dim(\ker FL_*)_x.$$

As a consequence, a local basis of $\mathscr{X}(S)^\perp$ is obtained from a corresponding one $\{\tilde{X}_\alpha\}$ of $\mathscr{X}(\mathscr{S}_S)^\perp$, taking a representative $X_\alpha \in \mathscr{X}(TQ)$ of every class $\tilde{X}_\alpha$ and adding a local

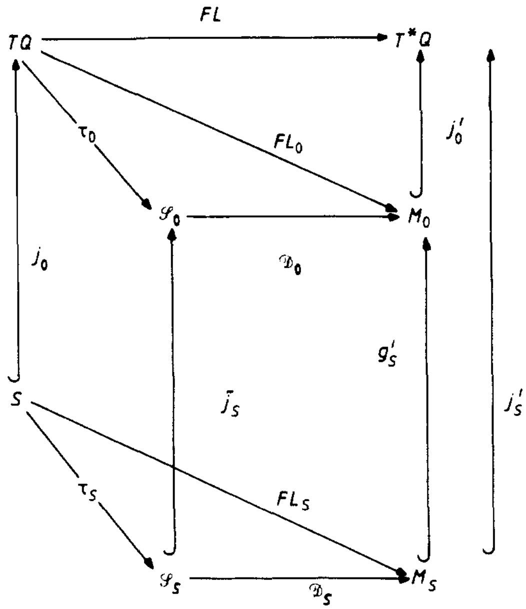

Figure 1.

basis $\{\Gamma_\mu\}$ of ker FL_*. Thus the desired local basis is (X_α, Γ_μ). Finally, since $\mathscr{D}_0$ and $\mathscr{D}_S$ are diffeomorphisms (figure 1), $\forall \tilde{X} \in \mathscr{X}(\mathscr{S}_S)^\perp$ one has

$$\tilde{j}_S^* i(\tilde{X})\tilde{\omega}_L = 0 \Leftrightarrow 0 = \mathscr{D}_S^{-1} * \tilde{j}_S^* i(\tilde{X})\tilde{\omega}_L = g_S^* \mathscr{D}_0^* i(\tilde{X})\tilde{\omega}_L = j_S'^* i(Z')\Omega$$

$$\Leftrightarrow Z' \in \mathscr{X}(M_S)^\perp \qquad \text{(where } Z'|_{M_0} = j'_{0*}\mathscr{D}_{0*}\tilde{X} \in \mathscr{X}(T^*Q)|_{M_0})$$

and simultaneously $Z' \in \boldsymbol{\mathscr{X}}(M_0)$ because it is the image of an FL-projectable vector field. In a similar way we can prove that, $\forall Z' \in \mathscr{X}(M_S)^\perp \cap \boldsymbol{\mathscr{X}}(M_0)$, the vector field $\tilde{X} \in \mathscr{X}(\mathscr{S}_0)$ such that $j'_{0*}\mathscr{D}_{0*}\tilde{X} = Z'|_{M_0}$ belongs to $\mathscr{X}(\mathscr{S}_S)^\perp$, and this concludes the proof of (i).

(ii) Taking into account that $\mathbf{ker}\,\boldsymbol{\omega}_{LS} = \mathscr{X}(S)^\perp \cap \boldsymbol{\mathscr{X}}(\mathscr{S})$ and $\mathscr{X}(TQ, S)_{FL} \supset \mathscr{X}(TQ)_{FL} \cap \boldsymbol{\mathscr{X}}(S)$, this assertion is a consequence of (i).

It is well known that, if (P, Ω) is a symplectic manifold and $j: S \hookrightarrow P$ is a submanifold, the canonical isomorphism $\hat{\Omega}: \mathscr{X}(P) \to \Lambda^1(P)$ allows one to associate a vector field $X_\zeta \in \mathscr{X}(S)^\perp$ to every constraint $\zeta \in C^0(P, S)$ as follows: $X_\zeta \equiv \hat{\Omega}^{-1}(\mathrm{d}\zeta)$ is the solution of $\mathrm{d}\zeta = i(X_\zeta)\Omega$. Conversely, there exists a local basis of $\mathscr{X}(S)^\perp$ made up of vector fields of this kind. Then, and since $X_\zeta(f) = \{f, \zeta\}$ (Poisson bracket) $\forall f \in \Lambda^0(P)$, the splitting in first and second class constraints [1] can be realised by studying whether or not $X_\zeta \in \boldsymbol{\mathscr{X}}(S)$ [15, 16]. As a consequence $\hat{\Omega}$ establishes an analogous correspondence between $\mathbf{ker}\,\boldsymbol{\omega}_S$ and the set of first class constraints.

These considerations, together with theorem 3.2, leads us to the following conclusions.

(i) Since $FL_*X \in \mathscr{X}(M_S)^\perp \cap \boldsymbol{\mathscr{X}}(M_0)$, $\forall X$ belonging to the FL-projectable basis of $\mathscr{X}(S)^\perp$, we have that FL_*X can be associated, by the canonical isomorphism in T^*Q, with constraints whose Poisson bracket with every primary constraint vanishes on M_0.

(ii) Since $FL_*X \in \mathbf{ker}\,\boldsymbol{\omega}'_S$, $\forall X$ belonging to the (weakly) FL-projectable basis of $\mathbf{ker}\,\boldsymbol{\omega}_{LS}$, we have that FL_*X can be associated with first class constraints on M_S.

(iii) In particular, $\forall X$ belonging to the FL-projectable basis of $\ker \omega_L$, $FL_* X$ can be associated with first class primary constraints relative to M_0 and, consequently, the second class ones have associated vector fields which do not have anti-images by FL in $\mathscr{X}(TQ)$ (since they are not tangent to M_0).

(iv) In addition, $\forall \zeta \in C^0(T^*Q, M_S)$ such that $X'_\zeta \in \mathscr{X}(M_0)$, we have that $FL^*\zeta = \chi \in C^0(TQ, S)$ (modulo primary constraints $C^0(T^*Q, M_0)$), and so

$$\mathrm{d}\chi = \mathrm{d}FL^*\zeta = FL^*\,\mathrm{d}\zeta = FL^* i(X'_\zeta)\Omega = i(X+\Gamma)\omega_L = i(X)\omega_L$$

where $X \in \mathscr{X}(S)^\perp / FL_* X = X'_\zeta$ and $\Gamma \in \ker FL_*$. Thus, an FL-projectable Lagrangian constraint can be associated with vector fields of $\mathscr{X}(TQ)$ which constitute a local basis of $\mathscr{X}(S)^\perp$ (although ω_L is a degenerate form).

4. On the *FL* projectability of constraint functions

It is evident that, given an FL-projectable constraint $\chi \in C^0(TQ, S)$, a non-FL-projectable equivalent constraint can be obtained by multiplying it by some non-FL-projectable function $f \notin \Lambda^0(TQ)_{FL}$. Nevertheless, there are non-FL-projectable constraints such that no FL-projectable equivalent constraint exists (see examples in [10]). These last constraints will be called strictly non-FL-projectable constraints. According to theorem 2.3 (i), one can prove that a necessary and sufficient condition for a non-FL-projectable constraint $\chi \in C^0(TQ, S)$ to be conversed in an equivalent FL-projectable one is that a function $f \in \Lambda^0(TQ)$ exists such that it is a solution of the system

$$f\Gamma_\mu(\chi) + \chi\Gamma_\mu(f) = 0 \qquad \forall \Gamma_\mu \in \ker FL_* / \Gamma_\mu(\chi) \neq 0$$

$$\Gamma_\nu(f) = 0 \qquad \forall \Gamma_\nu \in \ker FL_* / \Gamma_\nu(\chi) = 0.$$

The existence of strictly non-FL-projectable constraints is an intrinsic characteristic of a submanifold, as we shall prove in this section. The conclusion we will arrive at is that the presence of these constraints is equivalent to the fact that the submanifold defined by them cuts the foliation $\mathscr{F}^0_L$ in such a manner that, on each leaf, it eliminates as many degrees of freedom as the number of independent constraints of this kind that we have. This assertion is proved in the following way.

Theorem 4.1. Let (TQ, ω_L, E_L) be an almost regular Lagrangian system and $S \hookrightarrow TQ$ a submanifold. Then the number m_1 of independent strictly non-FL-projectable constraints contained in any base of $C^0(TQ, S)$ is equal to

$$m_1 = \dim(\ker FL_*)_x - \dim(\ker FL_* \cap \mathscr{X}(S))_x \qquad \forall x \in S$$

i.e. the number of independent vector fields of $\ker FL_*$ which are not tangent to S.

Proof. Let $m \equiv 2n - \dim S$ ($2n = \dim TQ = \dim T^*Q$) and $m_0 = \dim(\ker FL_*)_x$, $\forall x \in S$. Consider the submanifold $M_S \equiv FL(S)$. According to the discussion in paragraph (*b*) of the appendix, M_S is canonically identified with the quotient manifold $S/\mathscr{F}^S_L$($\mathscr{F}^S_L$ being the foliation induced in S by $\ker FL_{S*}$ or, equivalently, the part of $\ker FL_*$ tangent to S: $\ker FL_* \cap \mathscr{X}(S)$). Therefore, one has

$$\dim M_S = \dim S - \dim(\ker FL_* \cap \mathscr{X}(S))_x = 2n - m - m_0 + m_1.$$

On the other hand, denoting by m' the number of independent secondary constraints of $C^0(T^*Q, M_S)$ and by m_0 the number of independent primary ones (proposition A1 in the appendix), we have

$$\dim M_S = 2n - m_0 - m'$$

and hence $m_1 = m - m'$. But $FL^*(C^0(T^*Q, M_S)) = 0$ and, since $FL(S) = M_S$, m' is also the number of independent FL-projectable Lagrangian constraints of $C^0(TQ, S)$. Consequently m_1 is the number of independent strictly non-FL-projectable ones.

It is clear, after this theorem, that $m_1 < m_0$. Note that in the case when $\ker FL_* \cap \mathscr{X}(S) = \{0\}$ we have $C^0(TQ, S) \cap \Lambda^0(TQ)_{FL} = \varnothing$.

Another consequence of this theorem is that we can choose a basis of $C^0(TQ, S)$, $(\chi_\mu, \chi_{\mu'})$, and a local basis of $\ker FL_*$ $(\Gamma_\mu, \Gamma_{\mu'})$ $(\mu = 1, \ldots, m_1,\ \mu' = 1, \ldots, m - m_1)$ in such a manner that

$$\Gamma_\mu(\chi_\nu) = 0 \qquad \Gamma_\mu(\chi_{\mu'}) = 0 \qquad \forall \chi_\mu, \chi_{\mu'}, \forall \Gamma_\mu$$

$$\Gamma_{\mu'}(\chi_\nu) = 0 \qquad \det(\Gamma_{\mu'}(\chi_{\nu'}))_x \neq 0 \qquad \forall \chi_\nu, \chi_{\nu'}, \forall \Gamma_{\mu'}$$

and we assume that the last inequality is verified $\forall x \in S$. Therefore, it is evident that $\Gamma_\mu \in \ker FL_* \cap \mathscr{X}(S)$, but $\Gamma_{\mu'} \notin \mathscr{X}(S)$. Then a relation between the set of first class primary constraints (with respect to M_S) and the set of vector fields of $\ker FL_*$ which are not tangent to S can be established. This relation also extends to the set of strictly non-FL-projectable constraints (see [7]).

An immediate corollary of the last proposition is the following theorem.

Theorem 4.2. With the hypothesis of theorem 4.1, $\ker FL_* \subset \mathscr{X}(S)$ if and only if no base of $C^0(TQ, S)$ contains strictly non-FL-projectable constraints.

The preceding results suggest to us a way of characterising the properties of FL projectability of a submanifold $S \hookrightarrow TQ$. We can assign a number to every S, which indicates the maximal number of independent FL-projectable constraints of a base of $C^0(TQ, S)$. According to theorem 4.1, we can define this number geometrically as

$$\begin{aligned}\Upsilon_S &\equiv \dim TQ - \dim S - \dim(\ker FL_* \cap \mathscr{X}(S))_x \\ &= \dim T_x(TQ) - \dim T_xS - \dim(\ker FL_* \cap \mathscr{X}(S))_x \qquad \forall x \in S.\end{aligned}$$

It is obvious that, given two submanifolds $S \hookrightarrow S' \hookrightarrow TQ$, then $FL(S) = FL(S')$ if and only if $\Upsilon_S = \Upsilon_{S'}$ and, if $l = \dim S' - \dim S$, this is equivalent to demanding that

$$l = \dim(\ker FL_* \cap \mathscr{X}(S'))_x - \dim(\ker FL_* \cap \mathscr{X}(S))_x \qquad \forall x \in S.$$

5. *FL* projectability and the constraint algorithms

It is well known that, when we ask for the existence of consistent solutions of the equations of motion in the case of degenerate Lagrangians (i.e. solutions which are tangent to the submanifold where these equations are compatible), in the general case a sequence of submanifolds $TQ \hookleftarrow S_1 \hookleftarrow S_2 \hookleftarrow \ldots \hookleftarrow S_f$ is originated. The submanifold S_f where this sequence stabilises is called the final constraint submanifold. The same occurs in the Hamiltonian formalism: $T^*Q \hookleftarrow M_0 \hookleftarrow M_1 \hookleftarrow M_2 \hookleftarrow \ldots \hookleftarrow M_f$ (see [1, 4, 6, 7]) and, for almost regular Lagrangian systems, one has $M_k = FL(S_k)$, $\forall k$.

In the Lagrangian formalism we can distinguish two kinds of algorithms: the one taking as starting point the dynamical equations (1) without any other additional condition (called the presymplectic constraint algorithm (PCA) [3]), and the one which takes equations (1) together with the condition that their solutions be second-order differential equations (SODE) [8], i.e. the Euler–Lagrange equations [7]. In each case, the corresponding sequences of submanifolds are not the same. Indeed, in the first case we have the following.

Theorem 5.1. Let (TQ, ω_L, E_L) be an almost regular Lagrangian system. Then, if $P_k \hookrightarrow TQ$ is a submanifold of the sequence obtained from the PCA we have that

(i) there exists a basis of $C^0(TQ, P_k)$ made up of FL-projectable constraints whose FL projection constitutes a basis of $C^0(M_0, M_k)$; and

(ii) $\ker FL_* \subset \mathscr{X}(P_k)$.

Proof. For (i) see [4]. For (ii), an explicit proof is given in [5], although in this context it is a direct consequence of (i) and theorem 4.2.

On the other hand, the submanifolds obtained from the application of the algorithm for the Euler–Lagrange equations are locally defined, in general, by FL-projectable and strictly non-FL-projectable constraints [7]. Then, for any level k in both algorithms, we have $\Upsilon_{P_k} = \Upsilon_{S_k}$ and therefore $FL(P_k) = FL(S_k) = M_k$.

6. Conclusions

Summing up, the following remarks can be made.

(i) Starting from the usual concept of projectability by the application FL, a new weaker concept can be introduced when submanifolds of TQ and their images under FL have been considered. In the same way as the necessary and sufficient condition for the FL projectability of vector fields and p-forms is checked by studying their behaviour under the action of $\ker FL_*$, the necessary and sufficient condition for weakly FL projectability is tested by the part of $\ker FL_*$ which is tangent to the corresponding submanifold.

(ii) Typical vector fields characterising submanifolds of TQ are considered. The existence of local bases made up of FL-projectable or weakly FL-projectable vector fields (when they are tangent) is proved.

(iii) The presence of strictly non-FL-projectable constraints in any basis of $C^0(TQ, S)$ is associated with the existence of vector fields belonging to the non-tangent part of $\ker FL_*$. This allows one to label every submanifold with a number intrinsically defined, which informs us about the FL projectability of the constraints locally defining S.

(iv) Lagrangian constraint algorithms have been analysed. The relation between the existence of strictly non-FL-projectable constraints and the SODE condition becomes clear and, hence, it can be concluded that restrictions arising from the dynamical consistency of the equations of motion could lead to the elimination of degrees of freedom in TQ, but preserving the foliation $\mathscr{F}_L^0$, whereas restrictions imposed by the SODE condition could eliminate degrees of freedom on the leaves of $\mathscr{F}_L^0$ (see [4, 6]).

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Appendix. The distribution ker FL_*

Let us study some properties of the set ker FL_*. First, it is interesting to remark that ker $FL_* \subset \mathscr{X}(TQ)^{\text{v}}$ (vertical fields of $\mathscr{X}(TQ)$). In particular, ker $FL_* = \mathscr{X}(TQ)^{\text{v}} \cap \ker \omega_L$ [4, 6].

Ker FL_* is an involutive distribution and hence the Frobenius theorem ensures the existence of a regular foliation $\mathscr{F}_L^0$ of TQ generated by ker FL_* [5], in such a manner that all the points located on the same leaf $\mathscr{L}^0 \in \mathscr{F}_L^0$ (and only these ones) have the same image under FL, i.e. the fibres $FL^{-1}(FL(x))$, $\forall x \in TQ$, are the leaves of this foliation. This means that, given any point $x_0 \in TQ$, all the points having the same image by FL can be reached starting from it through the flux of the vector fields belonging to ker FL_*.

Let $\mathscr{S}_0 \equiv TQ/\mathscr{F}_{L^0}$ be the quotient space. Although the theory of reduction does not ensure that it is a manifold, the hypothesis that (TQ, ω_L, E_L) is an almost regular Lagrangian system suffices to claim that $\mathscr{S}_0$ is a manifold [5] and that the projection $\tau_0: TQ \to \mathscr{S}_0$ is a submersion. Therefore, $\mathscr{S}_0$ and M_0 can be canonically identified by means of a diffeomorphism $\mathscr{D}_0: \mathscr{S}_0 \to M_0$ (see figure 1). Then we have the following.

Proposition A1. The number of independent vector fields of ker FL_* is equal to $2n - \dim M_0$ (the number of independent primary constraints).

Proof. Immediate, because $\dim M_0 = \dim TQ - \dim(\ker FL_*)_x$, $\forall x \in TQ$.

Let $j_S: S \hookrightarrow TQ$ be a submanifold and $M_S = FL(S)$ with the embedding $j_{S'}: M_S \hookrightarrow T^*Q$. Let us introduce the map $FL_S: S \to M_S$, implicitly defined by $FL_0 j_S = j'_{S0} FL_S$, which is a submersion if FL is. We will distinguish two cases.

(*a*) *ker* $FL_* \cap \mathscr{X}(S) =$ *ker* FL_*. Since ker FL_* is an involutive distribution, we can state that ker $FL_*|_S$ gives rise to a regular foliation $\mathscr{F}_L^S$ of S. Then the vector fields $\Gamma_S \in \mathscr{X}(S)/j_{S*}\Gamma_S = \Gamma|_S \in \ker FL_*|_S$ are the elements of ker FL_{S*}. The quotient space $\mathscr{S}_S \equiv S/\mathscr{F}_L^S$ is a submanifold of $\mathscr{S}_0$, whose structure is inherited from the one of $\mathscr{S}_0$ [5]. It is isomorphically equivalent to M_S (see figure 1) and the projection $\tau_S: S \to \mathscr{S}_S$ is a submersion. This means that S is made up of the union of a subset of complete leaves of the foliation $\mathscr{F}_L^0$ of TQ. Furthermore, all the points of S located on the same leaf $\mathscr{L}^S \in \mathscr{F}_L^S$ (and only these ones) have the same image by FL (or FL_S).

(*b*) *ker* $FL_* \cap \mathscr{X}(S) \neq$ *ker* FL_*. With $\dim(\ker FL_* \cap \mathscr{X}(S))_x = m_1$, $\forall x \in TQ$. In this case, ker FL_* is made up of two kinds of fields: $\Gamma \in \mathscr{X}(S)$ and $\Gamma' \notin \mathscr{X}(S)$. Thus we have the following.

Proposition A2. The set $\{\Gamma_S\} \subset \mathscr{X}(S)/j_{S*}\Gamma_S = \Gamma|_S$, $\forall \Gamma \in \ker FL_* \cap \mathscr{X}(S)$, makes up ker FL_{S*}. Consequently, ker FL_{S*} is an involutive distribution.

Proof. Consider an extension $\mathbf{ker}\, \boldsymbol{FL_{S*}} \subset \mathscr{X}(TQ)$ of ker FL_{S*}. Then $\forall \Gamma \in \mathbf{ker}\, \boldsymbol{FL_{S*}}$, $\exists! \Gamma_S \in \ker FL_{S*}/j_{S*}\Gamma_S = \Gamma|_S$, and

$$FL_*\Gamma|_S = FL_* j_{S*}\Gamma_S = j'_{S*} FL_{S*}\Gamma_S = 0 \qquad \forall \Gamma|_S \in \ker FL_{*S}.$$

Conversely, $\forall\Gamma \in \ker FL_* \cap \mathfrak{X}(S)$, $\exists!\Gamma_S \in \mathscr{X}(S)/j_{S*}\Gamma_S = \Gamma|_S$ and so

$$0 = FL_*\Gamma|_S = FL_* j_{S*}\Gamma_S = j'_{S*}FL_{S*}\Gamma_S \Leftrightarrow FL_{S*}\Gamma_S = 0 \qquad \forall\Gamma_S \in \ker FL_{S*}$$

whence we conclude $j_{S*}(\ker FL_{S*}) = (\ker FL_* \cap \mathfrak{X}(S))|_S$.

We realise that S is not made up of a subset of complete leaves of $\mathscr{F}^0_L$ (as in the present case) but if we denote by $\mathscr{L}^S$ the leaves of the foliation $\mathscr{F}^S_L$ induced by ker FL_{S*} in S, we have $\mathscr{L}^S = \mathscr{L}'^0 \cap S$. Here $\{\mathscr{L}'^0\}$ is a subset of leaves of $\mathscr{F}^0_L$ whose union is a submanifold $P \hookrightarrow TQ$ having the properties $S \hookrightarrow P$ and $M_S = FL(S) = FL(P)$. Once again, only the points of S located on the same leaf have the same image by FL (or FL_S). The quotient space $S/\mathscr{F}^S_L \equiv \mathscr{S}_S$ is a submanifold of $\mathscr{S}_0$ isomorphically equivalent to M_S, and the projection $\tau_S : S \to \mathscr{S}_S$ is a submersion (see figure 1).

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